Math 245B Lecture 14 Notes

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1 Applications of the Baire Category Theorem in Banach Spaces

1.1 The complex Hahn-Banach theorem

Here is a loose end from last time.

Theorem 1.1 (Hahn-Banach, complex version)**.** *Let* $(\mathcal{X}, \|\cdot\|)$ *be a normed vector space over* $\mathbb{C}$*, let* $\mathcal{M} \subseteq \mathcal{X}$ *be a subspace, and let* $f \in \mathcal{M}^*$*. Then there exists an* $F \in \mathcal{X}^*$ *such that* $F|_{\mathcal{M}} = f$ *and* $|F| = f$*.*

Proof. Define $u = \operatorname{Re}(f)$. Observe that $f(ix) = if(x) = -\operatorname{Im}(f(x)) + i \operatorname{Re}(f(x))$. So $\operatorname{Im}(f) = -\operatorname{Re}(f(i\cdot)) = -u(i\cdot)$. By the real Hahn-Banach theorem, u extends to U, and let $F(x) = U(x) - iU(ix)$. Check that $|F| = |f|$. $\square$

1.2 The open mapping theorem

In finite dimensional vector spaces, linear bijections have linear inverses. Does this still work for normed spaces and bounded linear functions? The answer is no, unless we are dealing with Banach spaces.

Definition 1.1. A function $f : X \to Y$ is called **open** for all open $U \subseteq X$, $f[U]$ is open in Y.

Lemma 1.1. *Let* $T : \mathcal{X} \to \mathcal{Y}$ *be a linear map between normed spaces. Then* T *is open if and only if* $T[B_{\mathcal{X}}(0,1)] \supseteq B_y(0,r)$ *for some* $r > 0$*.*

Proof. ($\implies$): This follows from the definition.

($\impliedby$): Assume the condition holds. Let $U \subseteq \mathcal{X}$ be open, and let $x \in U$. Since U is open, there exists some $s > 0$ such that $B_{\mathcal{X}}(x,s) \subseteq U$. Then

$$\begin{aligned} T[U] &\supseteq T[B_{\mathcal{X}}(x,s)] \\ &= \{T(x+su) : u \in B_{\mathcal{X}}(0,1)\} \end{aligned}$$

$$
\begin{aligned}
&= Tx + sT[B_{\mathcal{X}}(0,1)] \\
&\supseteq Tx + sB_{\mathcal{Y}}(0,r) \\
&= B_y(Tx, sr).
\end{aligned}
$$

□

Theorem 1.2 (Open mapping theorem)**.** *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \to \mathcal{Y}$ be surjective. Then T is open.*

Proof. Step 1: Write $\mathcal{X} = \bigcup_n nB_{\mathcal{X}}(0,1)$. So $\mathcal{Y} = T[\mathcal{X}] = \bigcup_n nT[B_{\mathcal{X}}(0,1)]$. By the Baire category theorem, there is some $n \in \mathbb{N}$ such that $\overline{nT[B_{\mathcal{X}}(0,1)]}$ contains some open ball $B_{\mathcal{Y}}(y,r)$. Then $\overline{T[B_{\mathcal{X}}(0,1)]} \supseteq B_y(y/n, r_n)$. Pick x_1 such that $\|Tx_1 - y/n\| < r/(4n)$. Then $\overline{T[-x_1 + B_{\mathcal{X}}(0,1)]} = -Tx_1 + \overline{T[B_{\mathcal{X}}(0,1)]} \supseteq B_{\mathcal{Y}}(y/n - Tx_1, r/n) \supseteq B_{\mathcal{Y}}(0, r/(2n))$. So we get $\overline{T[B_{\mathcal{X}}(0, 1+\|x_1\|)]} \supseteq B_{\mathcal{Y}}(0, r/(2n))$. This gives us that $\overline{T[B_{\mathcal{X}}(0,1)]} \supseteq B_y(0,s)$ for some $s > 0$. By dilating by a constant (which is a homeomorphism from a Banach space to itself), we get $\overline{T[B_{\mathcal{X}}(0,r)]} \supseteq B_y(0,s)$ for all $r > 0$.

Step 2: Pick $y \in B_y(0,s)$, and pick $x_1 \in \mathcal{B}_{\mathcal{X}}(0,1)$ such that $\|y - Tx_1\|_{\mathcal{Y}} < s/2$. Call $y_1 = y - Tx_1$. Now pick $x_2 \in B_{\mathcal{X}}(0,1/2)$ such that $\|y_1 - T_{x_2}\|_{\mathcal{Y}} < s/4$, calling $y_2 = y_1 - T_{x_2}$. Continuing like this, we get a sequence $x_n \in B_{\mathcal{X}}(0, 1/2^{n-1})$ such that if $y_n = y_{n-1} - Tx_n$, then $\|y_n\| < s/2^n$. In the end, $x := \sum_n x_n \in \mathcal{X}$ as $\|x\| \le \sum_n \|x_n\| < 2$, and $Tx = \sum_n Tx_n = y$. So $T[B_{\mathcal{X}}(0,2)] \supseteq B_{\mathcal{Y}}(0, s/2)$. □

Corollary 1.1. *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a bijection between Banach spaces, then $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.*

Proof. T is open iff T^{-1} is continuous. □

Corollary 1.2. *If $\|\cdot\|_1 \le \|\cdot\|_2$ are 2 norms on $\mathcal{X}$ that are both complete, then $\|\cdot\|_1 \ge C\|\cdot\|_2$ for some C.*

Proof. Apply the previous corollary to $\mathrm{id} : (\mathcal{X}, \|\cdot\|_2) \to (\mathcal{X}, \|\cdot\|_1)$. □

1.3 The closed graph theorem

Definition 1.2. The **graph** of $T : \mathcal{X} \to \mathcal{Y}$ is $\Gamma(T) = \{(x, Tx) : x \in \mathcal{X}\} \subseteq \mathcal{X} \times \mathcal{Y}$.

If $\mathcal{T}$ is linear, then $\Gamma(T)$ is a subspace of $\mathcal{X} \times \mathcal{Y}$.

Theorem 1.3. *If $T : \mathcal{X} \to \mathcal{Y}$ linear between Banach spaces and $\Gamma(T)$ is a closed subspace of $\mathcal{X} \times Y$, then T is continuous.*

Remark 1.1. In general, if T is continuous, its graph is closed. If $\Gamma(T)$ is closed, T is called a **closed operator**.

Proof. Factorize T into $S(x) = (x, Tx)$ and $R(y, z) = z$.

$$\begin{tikzcd}
& \Gamma(T) \arrow[dr, "R"] & \\
\mathcal{X} \arrow[ur, "S"] \arrow[rr, "T"] & & \mathcal{Y}
\end{tikzcd}$$

$\Gamma(T)$ is closed, so it is a Banach space. R is continuous, so it suffices to show that S is continuous. But S is a bijection, and $S^{-1} : (y, z) \to y$ is continuous, so the open mapping theorem implies that S is continuous. $\square$

Why do we care? Continuous means that if $x_n \to x$, then $Tx_n \to Tx$. To show that something has a closed graph, we only need to show that if $(x_n, Tx_n) \to (x, y)$, then $y = Tx$. So we don't need to show that such an x exists; we only need to show that if it does, then x_n converges to the right thing.

1.4 The uniform boundedness principle

Theorem 1.4 (uniform boundedness principle)**.** *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, $\mathcal{X}$ be Banach, and let $\mathcal{A} \subseteq L(\mathcal{X}, \mathcal{Y})$. If $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all $x]in\mathcal{X}$, then $\sup_{T \in \mathcal{A}} \|T\| < \infty$.*

Proof. Let $E_n = \{x \in \mathcal{X} : \sup_{T \in \mathcal{A}} \|Tx\| \leq n\} = \bigcap_{T \in \mathcal{A}} \{x : \|Tx\| \leq n\}$. Then E_n is closed, and $\mathcal{X} = \bigcup_n E_n$, so by Biare category, $E_{n/r} \supseteq B_{\mathcal{X}}(x, 1)$ for some n, x, r. Then $B_{\mathcal{X}}(x, 1) - B_{\mathcal{X}}(x, 1) \subseteq E_{2n/r}$. But the left hand side contains $B_{\mathcal{X}}(0, 2)$. So $\|x\| < 2 \implies \|Tx\| \leq 2n/r$ for all $T \in \mathcal{A}$. This is independent of x, so $\|T\| \leq n/r$ for all $T \in \mathcal{A}$ $\square$